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Lattice Gas Generalization of the Hard Hexagon Model. I. Star-Triangle Relation and Local Densities

R. J. Baxter[1] **and George E. Andrews**[2]

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In the solvable hard hexagon model there is at most one particle in every pair of adjacent sites, and the solution automatically leads to various mathematical identities, in particular to the Rogers–Ramanujan relations. These relations have been generalized by Gordon. Here we construct a solvable model with at most two particles per pair of adjacent sites, and find the solution involves the next of Gordon's relations. We conjecture the corresponding solution for a model with at most n particles per pair of adjacent sites: this involves all Gordon's relations, as well as others that we will discuss in a subsequent paper.

KEY WORDS: Statictical mechanics; lattice gas; star-triangle relation; Yang–Baxter relation.

1. INTRODUCTION

The hard hexagon model was solved[1] in 1979 by generalizing it to a model which satisfied the "star-triangle" (or "Yang–Baxter") relations. Its local density and order parameters could then be calculated by the corner transfer method.[2,3] An intriguing aspect of the solution was that these properties were obtained as series of the type that occurs in the theory of partitions,[4] and could be simplified to elliptic θ-function-type products by using the Rogers–Ramanujan and related identities. This simplification made it easy (by using the conjugate modulus identities of the elliptic functions) to obtain the behavior at the critical point.

[1] Research School of Physical Sciences, The Australian National University, G.P.O. Box 4, Canberra 2601, Australia.

[2] Department of Mathematics, McAllister Building, Pennsylvania State University, University Park, Pennsylvania 16802, U.S.A.

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The Rogers–Ramanujan relations have been generalized by Gordon,[4,5] in which form they can be written as

$$\sum_{\sigma_2, \sigma_3, \ldots} q^{\sigma_2 + 2\sigma_3 + 3\sigma_4 + \cdots} = \prod_{\substack{j=1 \\ j \neq 0, \pm i (\mathrm{mod}\ 2n+1)}}^{\infty} \tag{1.1}$$

Here i, n, σ_1, σ_2, σ_3,... are nonnegative integers, subject to the constraints

$$1 \leqslant i \leqslant n \qquad \sigma_1 = n - i \tag{1.2}$$

$$0 \leqslant \sigma_j \leqslant n-1 \qquad 0 \leqslant \sigma_j + \sigma_{j+1} \leqslant n-1 \ \forall j \geqslant 1 \tag{1.3}$$

q is any complex number such that $|q| < 1$; the sum in (1.1) is over all values of σ_2, σ_3,... satisfying (1.3).

For $n = 2$ we regain the original Rogers–Ramanujan identities. In this case the left-hand side (LHS) of (1.1) occurs in the hard hexagon model calculations, σ_1, σ_2,... being the occupation numbers of a line of sites on the lattice. The constraint (1.3) corresponds to the requirement that adjacent sites cannot both contain a particle.

This leads one to speculate that there may be a solvable generalization of the hard hexagon model that yields the LHS of (1.1) for arbitrary n. One would now place particles on the sites of the square lattice, subject to the rule that there be no more than $n-1$ particles on each pair of adjacent sites. (This implies that each site contains no more than $n-1$ particles.) It would be an "interactions-round-a-face" (IRF) model, with interactions between the fair sites round each face.

A number of solutions of the star-triangle relations have been found in recent years,[6–10] but none are of this type. In particular, the eight-vertex SOS model[7] did yield generalizations of the Rogers–Ramanujan identities, indeed it even gave expressions that were identical with the RHS of (1.1). We thought we were close to our goal, and would soon discover a mapping from the 8V SOS model to the required particle model.

In fact we have not succeeded in doing this. Instead we here obtain directly the solvable particle model for the case $n = 3$. Like the hard hexagon model and the 8V SOS model, it has four regimes, depending on the values of its parameters. In one of them (Regime I) the local densities are proportional to the LHS of (1.1) as we hoped. In the other regimes we obtain similar, but more complicated, expressions.

We have not constructed the particle model for $n > 3$; but, from the now-known $n = 2$ and 3 cases, we have conjectured the local density expressions for arbitrary n. In a subsequent we show, for all regimes and for arbitrary n, that there exist identities analogous to (1.1) that enable us to write the results in terms of elliptic θ-function-type products.

2. STAR-TRIANGLE RELATION

Consider a square lattice $\mathscr{L}$ of N sites. Place particles on the sites and let $\sigma_i(=0, 1, 2,...)$ be the number of particles at site i. Let i, j, k, l be the four sites round a face (arranged as in Fig. 1) and define a Boltzmann weight function $w(\sigma_i, \sigma_j\ \sigma_k, \sigma_l)$ for each face. Then the partition function is

$$z = \sum_{\sigma} \prod_{(ijkl)} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{2.1}$$

where the product is over all faces $(ijkl)$ of $\mathscr{L}$ and the sum is over all allowed values of $\sigma \equiv \{\sigma_1,..., \sigma_N\}$. For a given function w, we want in statistical mechanics to calculate the partition function per site

$$z = \lim_{N \to \infty} Z^{1/N} \tag{2.2}$$

where the limit is that in which $\mathscr{L}$ becomes large in all directions. We are also interested in the probability that a particle site, say site 1, contains r particles, which

$$P_r = Z^{-1} \sum_{\sigma} \delta(\sigma_1, r) \prod_{(ijkl)} w(\sigma_i, \sigma_j, \sigma_k, \sigma_l) \tag{2.3}$$

it being understood that $\mathscr{L}$ is again large, site 1 being deep in its interior.

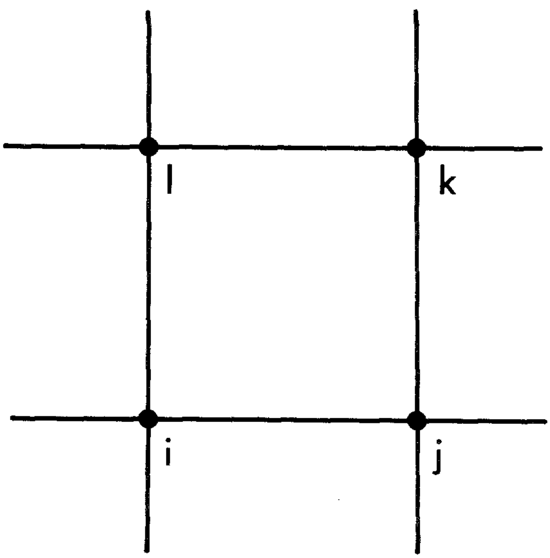

Fig. 1. A typical face of the square lattice, with surrounding sites i, j, k, l.

The quantities z and P_r can be evaluated[3] if, given w, one can find a family of nontrivial solutions for w', w'' of the star-triangle or Yang–Baxter relation

$$\sum_g w(a, b, g, f)\, w'(e, f, g, d)\, w''(c, d, g, b) = \sum_g w''(g, e, f, a)\, w'(g, a, b, c)\, w(g, c, d, e) \tag{2.4}$$

for all values of $a, b, \ldots, f$. (An example of a trivial solution is $w'(a, b, c, d) = 2w(b, c, d, a)$, $w''(a, b, c, d) = \delta(a, c)$.)

One obtains P_r by the corner transfer matrix method[2,3,7], which gives

$$P_r = \sum_{\sigma_1,\ldots,\sigma_m} \delta(\sigma_1, r)\, M(\sigma_1,\ldots,\sigma_m) \Big/ \sum_{\sigma_1,\ldots,\sigma_m} M(\sigma_1,\ldots,\sigma_m) \tag{2.5}$$

Here the $M(\sigma_1, \sigma_2,\ldots, \sigma_m)$ (for all values of $\sigma_1,\ldots, \sigma_m$) are the eigenvalues of the product of the four-corner transfer matrices. We can identify $\sigma_1,\ldots, \sigma_m$ with the spins on a horizontal line of sites of the lattice $\mathscr{L}$, starting at the centre (site 1) and going out to the boundary (site m). The large-lattice limit implies that we ultimately let m become infinite. If we define

$$F(\sigma_1) = \sum_{\sigma_2,\ldots,\sigma_m} M(\sigma_1, \sigma_2,\ldots, \sigma_m) \tag{2.6}$$

then (2.5) implies

$$P_r = F(r) \Big/ \sum_{\sigma_1 \geqslant 0} F(\sigma_1) \tag{2.7}$$

so $F(r)$ can be regarded as an unnormalized probability.

The star-triangle relation (2.4) has solutions only for certain special functions w (corresponding to "solvable" models). Here we are interested in obtaining solutions when the constraints corresponding to (1.3) are satisfied, i.e.

$$0 \leqslant \sigma_j \leqslant n-1 \qquad 0 \leqslant \sigma_i + \sigma_j \leqslant n-1 \tag{2.8}$$

for all edges $\langle ij \rangle$ of $\mathscr{L}$. This means that we can take, for a, b, c, d non-negative integers

$$w(a, b, c, d) = 0 \qquad \text{unless} \quad a+b,\ b+c,\ c+d.\ d+a < n \tag{2.9}$$

We impose the diagonal–reflection symmetry conditions

$$w(a, b, c, d) = w(c, b, a, d) = w(a, d, c, b) \tag{2.10}$$

Since $\sigma_1,\ldots, \sigma_m$ in (2.5) and (2.6) correspond to a linear array of sites, they must satisfy the constraints (1.3).

3. THE CASE $n=2$ (HARD HEXAGONS)

For $n=2$ the function $w(a, b, c, d)$ is known, being that of the generalized hard hexagon model. For completeness we give the results here.

First let $\theta_1(u, p)$, or $\theta_1(u)$, be the elliptic θ-function

$$\theta_1(u, p)=2p^{1/8}\sin u \prod_{j=1}^{\infty}(1-2p^j\cos 2u+p^{2j})(1-p^j) \tag{3.1}$$

where p is real and $|p|<1$, and let

$$\lambda=\pi/5 \tag{3.2}$$

Then, apart from an overall normalization factor, $w(a, b, c, d)$ is given by

$$\begin{aligned}
&w(0,0,0,0)=\theta_1(2\lambda+u)/\theta_1(2\lambda)\\
&w(1,0,0,0)=w(0,0,1,0)=x^{-1}\theta_1(u)/[\theta_1(\lambda)\,\theta_1(2\lambda)]^{1/2}\\
&w(0,1,0,0)=w(0,0,0,1)=x\theta_1(\lambda-u)/\theta_1(\lambda)\\
&w(1,0,1,0)=x^{-2}\theta_1(2\lambda-u)/\theta_1(2\lambda)\\
&w(0,1,0,1)=x^2\theta_1(\lambda+u)/\theta_1(\lambda)
\end{aligned} \tag{3.3}$$

(eq. 14.2.39 of Ref. 3).

This function satisfies the symmetry relations (2.10), i.e.

$$w(a, b, c, d)=w(c, b, a, d)=w(a, d, c, b) \tag{3.4}$$

It depends on the parameters u, x and p. We regard p as a given constant, u and x as variables, and write w as $w(a, b, c, d\,|\,u, x)$, or simply $w[u, x]$. Then it has the rotation and inversion symmetries

$$w(a, b, c, d\,|\,u, x)=w(b, c, d, a\,|\,\lambda-u, x_0/x) \tag{3.5}$$

$$\sum_g w(a, b, g, d\,|\,u, x)\,w(g, b, c, d\,|\,-u, x^{-1})=\delta(a, c)\,\theta_1(\lambda+u)\,\theta_1(\lambda-u)/\theta_1^2(\lambda) \tag{3.6}$$

where

$$x_0=[\theta_1(\lambda)/\theta_1(2\lambda)]^{1/2} \tag{3.7}$$

The function w simplifies when $u=0$ or λ, these being inversion points.[11] It then has the values

$$w(a, b, c, d\,|\,0, x)=x^{b+d-a-c}\,\delta(a, c) \tag{3.8}$$

$$w(a, b, c, d\,|\,\lambda, x)=(x_0/x)^{a+c-b-d}\,\delta(b, d) \tag{3.9}$$

If we define τ (a complex number) by

$$p = e^{2\pi\tau i} \tag{3.10}$$

then the function $\theta_1(u)$ has the quasi-periodicity properties

$$\theta_1(u) = -\theta_1(u+\pi) = -p^{1/2}e^{2iu}\theta_1(u+\tau\pi) \tag{3.11}$$

It follows that

$$w[u, x] = -w[u+\pi, x] = p^{1/2}e^{i(2u-\lambda)}w[u+\tau\pi, -xe^{-i\lambda}] \tag{3.12}$$

The x dependence is rather trivial

$$w(a, b, c, d \mid u, x) = x^{b+d-a-c}w(a, b, c, d \mid u, 1) \tag{3.13}$$

Using these properties (3, 4)-(3.13), we can show that the star-triangle relation (2.4) is satisfied by

$$w = w[u, x] \qquad w' = w[u', x'] \qquad w'' = w[u'', x''] \tag{3.14}$$

provided only that the six variables u, u', u'', x, x', x'' satisfy the relations

$$u + u' + u'' = \lambda \qquad xx'x'' = x_0 \tag{3.15}$$

To prove this, consider the difference of the LHS and RHS of (2.4) for given values of $a, ..., f$. From (3.13) it is easily found that

$$\text{LHS} - \text{RHS} = x^{-a-d}x'^{-b-e}x''^{-c-f}\Xi \tag{3.16}$$

where Ξ depends on x, x', x'' only via their product $xx'x''$. It also depends on u, u', u''. If we regard λ and u' as fixed, u'' and $xx'x''$ as given by (3.15), i.e., $u'' = \lambda - u' - u$, then Ξ is a function only of u

$$\Xi \equiv \Xi(u) \tag{3.17}$$

Incrementing u by $\tau\pi$ causes u'' to be decreased by $\tau\pi$. From (3.12) and (3.14), the effect on w and w'' is to multiply them by certain factors and to replace x, x'' by $-xe^{i\lambda}$, $-x''e^{-i\lambda}$, respectively. This leaves $xx'x''$ unchanged, so (3.15) remains satisfied and we obtain

$$\Xi(u+\tau\pi) = p^{-1}e^{2i(u''-u)}e^{i(\lambda-\pi)(c+f-a-d)}\Xi(u) \tag{3.18a}$$

It is also easily seen that

$$\Xi(u+\pi) = \Xi(u) \tag{3.18b}$$

and that $\Xi(u)$ is an entire function of u. It follows that there exist constants C, u_1, u_2 (independent of u but dependent on $a,..., f$) such that

$$\Xi(u) = C\theta_1(u-u_1)\,\theta_1(u-u_2) \tag{3.19}$$

where, to modulo π

$$u_1 + u_2 = \lambda - u' + \tfrac{1}{2}(\lambda - \pi)(c + f - a - d) \tag{3.20}$$

Since $\theta_1(0) = 0$, u_1 and u_2 are zeros of $\Xi(u)$.

Now consider the case when $u = \lambda - u'$, and so $u'' = 0$. Then from (3.5) and (3.11)

$$w'(a, b, c, d) = (x_0/xx')^{a+c-b-d} w(b, c, d, a) \tag{3.21}$$

while from (3.8)

$$w''(a, b, c, d) = x''^{b+d-a-c}\,\delta(a, c) \tag{3.22}$$

Substituting these expressions into (2.4), we readily find, using (3.15), that it is satisfied. (In fact we have a variant of the trivial solution mentioned immediately after (2.4).) Thus

$$\Xi(\lambda - u') = 0 \tag{3.23}$$

Next take $u = -u'$, i.e., $u'' = \lambda$. Since Ξ in (3.16) involves x, x', x'' only via $xx'x''$, we can, without loss of generality, take $x'' = x_0$ and $x' = x^{-1}$. Then from (3.9) we have $w''(a, b, c, d) = \delta(b, d)$. Using (3.4), we can write (2.4) as

$$\begin{aligned} &\delta(b, d) \sum_g w(a, b, g, f \mid u, x)\, w(g, b, e, f \mid -u, x^{-1}) \\ &\quad = \delta(a, e) \sum_g w(b, c, g, a \mid -u, x^{-1})\, w(g, c, d, a \mid u, x) \end{aligned} \tag{3.24}$$

The g sums can at once be evaluated from the inversion relation (3.6). Both sides become $\delta(b, d)\,\delta(a, e)$, and so (2.4) is satisfied in this case and

$$\Xi(-u') = 0 \tag{3.25}$$

Cyclically permuting u, u', u'' (and x, x', x'') is equivalent to permuting a, c, e and b, d, f in (2.4). Thus $\Xi(u)$ vanishes if any of u, u', u'' equal 0 or λ. In particular

$$\Xi(\lambda - u') = \Xi(-u') = \Xi(0) = \Xi(\lambda) = 0 \tag{3.26}$$

Thus $\Xi(u)$ has four zeros, in general, distinct to moduli π and $\tau\pi$, while from (3.19) it either has only two, or the constant C vanishes. It follows that C must vanish, so $\Xi(u)$ is identically zero and hence (2.4) is satisfied.

(This argument is a generalization of the theorem that if a polynomial of degree n has more than n distinct zeros, then it vanishes identically.)

4. THE CASE $n=3$; SOLUTION OF THE STAR-TRIANGLE RELATION

We can of course attempt to solve the star-triangle relation directly, and for $n=2$ this is how the hard hexagon model was solved.[1,3] From (2.9) and (2.10), each function w, w', w'' has five independent values and (2.4) yields seven distinct constraints. Thus one has seven equations for 15 unknowns, and one can attempt to parametrize the solution.

For $n=3$ the situation is much worse: each of w, w', w'' has 15 independent values, and (2.4) yields 59 distinct constraints, so one has 59 equations for 45 unknowns.

Jimbo and Miwa[8] have obtained other solutions of the star-triangle relations by using the differential formulation.[12] We were unable to make progress by this method, but finally managed to obtain w, w', w'' by first using the simpler of the 59 constraints (see Appendic A). For brevity, it is convenient to often abbreviate the θ-function $\theta_1(u)$, defined by (3.1), simply to (u). With this convention, the result for $w(a, b, c, d)$ is

$$\begin{aligned}
w(1100) &= \pm x_0^{-1} y_0 (u)(\lambda-u)/(\lambda)(3\lambda)\\
w(1000) &= x_0 x^{-1}(u)(3\lambda+u)/(\lambda)(3\lambda)\\
w(2000) &= y_0 y^{-1}(u)(3\lambda-u)/(\lambda)(2\lambda)\\
w(0111) &= x_0^{-1} x(u)(2\lambda+u)/(\lambda)(3\lambda)\\
w(1020) &= x_0 y_0 x^{-1} y^{-1}(u)(\lambda+u)/(\lambda)(2\lambda)\\
w(0100) &= x(\lambda-u)(3\lambda+u)/(\lambda)(3\lambda)\\
w(0200) &= y(\lambda-u)(2\lambda+u)/(\lambda)(2\lambda)\\
w(1110) &= x^{-1}(\lambda-u)(3\lambda-u)/(\lambda)(3\lambda)\\
w(0201) &= xy(\lambda-u)(2\lambda-u)/(\lambda)(2\lambda)\\
w(1010) &= x^{-2}(3\lambda+u)(3\lambda-u)/(3\lambda)^2\\
w(0101) &= x^2(3\lambda+u)(2\lambda+u)/(2\lambda)(3\lambda)\\
w(2020) &= y^{-2}(2\lambda-u)(3\lambda-u)/(2\lambda)(3\lambda)\\
w(0202) &= y^2(\lambda+u)(2\lambda+u)/(\lambda)(2\lambda)\\
w(1111) &= (2\lambda+u)(3\lambda-u)/(2\lambda)(3\lambda)\\
w(0000) &= w(1111)-(2\lambda)(u)(\lambda-u)/(\lambda)(3\lambda)^2
\end{aligned} \tag{4.1}$$

where u, x, y, p are arbitrary variables, x_0 and y_0 are given by

$$x_0 = [(2\lambda)/(3\lambda)]^{1/2} \qquad y_0 = [(\lambda)/(3\lambda)]^{1/2} \tag{4.2}$$

and now instead of (3.2) we have

$$\lambda = \pi/7 \tag{4.3}$$

Together with (2.10), these equations define the weight function $w(a, b, c, d)$.

Analogously to Section 3, we regard p as constant, u, x, and y as variables, and write w as $w(a, b, c, d \mid u, x, y)$ or simply $w[u, x, y]$. Then (2.4) is satisfied by

$$w = w[u, x, y] \qquad w' = w[u', x', y'] \qquad w'' = w[u'', x'', y''] \tag{4.4}$$

provided that

$$u + u' + u'' = \lambda \qquad xx'x'' = x_0 \qquad yy'y'' = y_0 \tag{4.5}$$

One way of verifying this assertion would be to check each of the 59 equations (2.4). This would be extremely tedious: instead we start by generalizing the method of Section 3.

Define ϕ_a by

$$\phi_0 = 1 \qquad \phi_1 = x \qquad \phi_2 = y \tag{4.6}$$

then the analogues of the properties (3.5)–(3.13) are

$$\begin{aligned} w(a, b, c, d \mid u, x, y) = w(b, c, d, a \mid \lambda - u, x_0/x, y_0/y) \\ \times \sum_g w(a, b, g, d \mid u, x, y) \\ \times w(g, b, c, d \mid -u, x^{-1}, y^{-1}) \end{aligned} \tag{4.7}$$

$$= \delta(a, c)(\lambda + u)(\lambda - u)(2\lambda + u)(2\lambda - u)/[(\lambda)^2(2\lambda)^2] \tag{4.8}$$

$$w(a, b, c, d \mid 0, x, y) = \phi_b \phi_d \delta(a, c)/[\phi_a \phi_c] \tag{4.9}$$

$$w(a, b, c, d \mid \lambda, x_0/x, y_0/y) = \phi_a \phi_c \, \delta(b, d)/[\phi_b \phi_d] \tag{4.10}$$

$$w[u, x, y] = w[u + \pi, x, y] \tag{4.11a}$$

$$= p e^{2i(2u - \lambda)} w[u + \tau\pi. \ -xe^{-i\lambda}, \ -ye^{-3i\lambda}] \tag{4.11b}$$

$$w(a, b, c, d \mid u, x, y) = \phi_b \phi_d w(a, b, c, d \mid u, 1, 1)/[\phi_a \phi_c] \tag{4.12}$$

These properties follow straightforwardly from (4.1); the most tedious to establish is the inversion relation (4.8), where one makes repeated use of the identities

$$(u) = (7\lambda - u) = -(-u) \tag{4.13}$$

$$\frac{(r+u)(r-u)}{(r)^2} - \frac{(r+s)(r-s)(u)^2}{(r)^2(s)^2} = \frac{(s+u)(s-u)}{(s)^2} \tag{4.14}$$

true for all complex numbers u, r, s (c.f. eq. 15.3.10 of Ref. 3). A particular case that occurs frequently is when $u, r, s = \lambda, 3\lambda, 2\lambda$

$$\frac{(2\lambda)}{(3\lambda)} - \frac{(\lambda)^3}{(2\lambda)(3\lambda)^2} = \frac{(\lambda)(3\lambda)}{(2\lambda)^2} \tag{4.15}$$

Now consider the difference of the LHS and RHS of (2.4) for given values of $a, \ldots, f$. Let $\phi'_n[\phi''_a]$ be obtained from definition (4.6) of $\mathscr{D}_a$ by replacing x, y by $x'y'[x'', y'']$. Then from (4.12)

$$\text{LHS} - \text{RHS} = \Xi / [\phi_a \phi_d \phi'_b \phi'_e \phi''_c \phi''_f] \tag{4.16}$$

where Ξ depends on x, x', x'', y, y', y'' only via the function product $\phi_a \phi'_a \phi''_a$, i.e., only via $xx'x''$ and $yy'y''$. As in Section 3, we regard λ and u' as fixed, u'', $xx'x''$ and $yy'y''$ as given by (4.5). Then Ξ is a function only of u, as in (3.17). From (4.11), it satisfies the quasi-periodicity relations

$$\Xi(u + \tau\pi) = p^{-2} e^{4i(u'' - u)} e^{i(\lambda - \pi)(\nu_c + \nu_f - \nu_a - \nu_d)} \Xi(u) \tag{4.17a}$$

$$\Xi(u + \pi) = \Xi(u) \tag{4.17b}$$

where

$$\nu_a = a(a+1)/2 \tag{4.18}$$

Since $\Xi(u)$ is entire, it follows that there exist constants C, u_1, u_2, u_3, u_4 such that

$$\Xi(u) = C\theta_1(u - u_1)\,\theta_1(u - u_2)\,\theta_1(u - u_3)\,\theta_1(u - u_4) \tag{4.19}$$

$$u_1 + u_2 + u_3 + u_4 = 2(\lambda - u') + \tfrac{1}{2}(\lambda - \pi)(\nu_c + \nu_f - \nu_a - \nu_d) \tag{4.20}$$

Now look at special values of u. When $u = \lambda - u'$ we have $u'' = 0$. Using (4.9) and (4.7) we easily find that (2.4) is satisfied, so $\Xi(\lambda - u') = 0$. Next we take $u = -u'$, i.e., $u'' = \lambda$, $x'' = x_0$, $y'' = y_0$, $x' = x^{-1}$, and $y' = y^{-1}$. Then from (4.10) we have $w''(a, b, c, d) = \delta(b, d)$, so (2.4) takes a form very similar to (3.24), but with the single argument x replaced by x, y [and x^{-1} by x^{-1}, y^{-1}]. From the inversion relation (4.8) it follows that (2.4) is satisfied, and hence $\Xi(-u') = 0$.

Thus again we have (3.23) and (3.25) and therefore (3.26). There are two ways of reconciling (3.26) with (4.19): either $C=0$ or

$$u_1,\dots, u_4 = \lambda - u', \qquad -u', 0, \lambda$$

In the latter case, it follows from (4.20) that

$$v_c + v_f = v_a + v_d \tag{4.22}$$

Thus if $a,\dots, f$ do *not* satisfy (4.22), then C must be zero and we have proved that (2.4) is satisfied for all complex numbers u.

We can of course apply these arguments as well to u' and u'' as to the variable u. Since cyclically permuting u, u', u'' is equivalent to permuting a, c, e and b, d, f in (2.4), it follows that we have verified (2.4), except only for the cases when

$$v_a + v_d = v_b + v_e = v_c + v_f \tag{4.23}$$

In (2.4) the spins a, b are nonnegative integers satisfying the constraint $a+b \leqslant 2$. Similarly for the pairs (b, c), (c, d), (d, e), (e, f), and (f, a). Interchanging a with d, b with e, and c with f merely interchanges the left and right sides of the equation. It follows that there are only five distinct equations for which (4.23) is satisfied, namely those with $(a,\dots, f)$ equal to $(0, 1, 0, 1, 0, 1)$, $(0, 2, 0, 2, 0, 2)$, $(1, 1, 0, 0, 0, 1)$, $(1, 1, 1, 0, 0, 0)$, $(0, 1, 1, 1, 0, 0)$. The fourth and fifth can be obtained from the third by cyclic permutations of u, u', u'', so it is sufficient to consider the first three equations.

It is easily verified that these first three equations are satisfied for $u = -3\lambda, -\lambda, -2\lambda$, respectively. (In each case we just have to prove that a product of θ-functions is the same as another product.) Thus for these values of $a,\dots, f$, $\Xi(u)$ has one other zero, in addition to those given by (3.26). It therefore has more than four zeros (distinct to moduli π, $\tau\pi$), so the constant C in (4.19) must vanish. Hence $\Xi(u)$ vanishes identically for *all* values of $a,\dots, f$, i.e., the star-triangle relations (2.4) are satisfied.

5. CORNER TRANSFER MATRICES: $n=3$

Let A, B, C, D be the four corner transfer matrices[2,3,7,13], corresponding, respectively, to the lower-right, upper-right, upper-left, and lower-left quadrants, as in Fig. 13.2 of Ref. 3. They have elements $A_{\sigma\sigma'},\dots, D_{\sigma\sigma'}$, where $\boldsymbol{\sigma} = \{\sigma_1,\dots, \sigma_m\}$ is a line of spins radiating out from the center of the lattice, σ_1 being the central spin σ_m, a boundary spin. The elements are zero unless

$\sigma_1 = \sigma'_1$, which means that $A,..., D$ are all block-diagonal matrices, commuting with the diagonal matrix R that has elements

$$R_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} = r_{\sigma_1} \prod_{j=1}^{m} \delta(\sigma_j, \sigma'_j) \tag{5.1}$$

for arbitrary choices of r_a.

Consider the $n=3$ case discussed in Section 4. Let u (and x and y) be different in each quadrant, having the values $u_1,..., u_4$ ($x_1,..., x_4$ and $y_1,..., y_4$) in the four quadrants, respectively. Then we can write A as $A(u_1, x_1, y_1)$, B as $B(u_2, x_2, y_2)$, etc. Because the star-triangle relation (2.4) is satisfied by (4.4) and (4.5), A, B, C, D satisfy various product and commutation relations. We have to distinguish two domains in the complex u plane

$$\begin{aligned} \mathscr{D}_1 &: 0 < \mathrm{Re}(u) < \lambda \\ \mathscr{D}_2 &: \lambda - \tfrac{1}{2}\pi < \mathrm{Re}(u) < 0 \end{aligned} \tag{5.2}$$

Provided $u_1,..., u_4$ all lie in the same domain, we can establish that (to within an overall irrelevant scalar normalization factor)

$$\begin{aligned} &A(u_1, x_1, y_1)\, B(u_2, x_2, y_2)\, C(u_3, x_3, y_3)\, D(u_4, x_4, y_4) \\ &\quad = PRM \exp[(u_1 - u_2 + u_3 - u_4)\, \mathscr{H}]\, P^{-1} \end{aligned} \tag{5.3}$$

Here P, M, $\mathscr{H}$ are matrices that are independent of $u_1,..., u_4$, $x_1,..., x_4$, and $y_1,..., y_4$ (but depend on the choice of domain); R, M, $\mathscr{H}$ are diagonal, R being given by (5.1) with

$$r_0 = 1 \qquad r_1 = x_1 x_3 / x_2 x_4 \qquad r_2 = y_1 y_3 / y_2 y_4 \tag{5.4}$$

For the regular case, when u_i, x_i, y_i are independent of i, the RHS of (5.3) reduces to PMP^{-1}. Thus M is the diagonal matrix of eigenvalues of $ABCD$, its elements being the $M(\sigma_1,..., \sigma_m)$ needed in (2.5). Our aim here is to calculate M (but not P). We need only calculate M to within a scalar factor, because such factors cancel out of (2.5).

We start by relating M to $\mathscr{H}$. First let $u_1, u_3 \to 0$, $x_1, x_3, y_1, y_3 = 1$. Then from (4.9), $w(a, b, c, d) = \delta(a, c)$ in the first and third quadrants, which implies that $A = C = I$ (the identity matrix).

Now use the rotation relation (4.7) in the inversion relation (4.8). The result is an inversion relation between $w[\lambda - u, x_0/x, y_0/y]$ and $w[\lambda + u, x_0 x, y_0 y]$. In domain $\mathscr{D}_1$ this can be applied directly to $B(u, x, y)$

and $D(u, x, y)$ (one has to analytically continue across the $\mathrm{Re}(u) = \lambda$ boundary, but this is usual for inversion relations), yielding

$$B(\lambda - u, x_0/x, y_0/y)\, D(\lambda + u, x_0 x, y_0 y) = I \tag{5.5}$$

(to within a scalar factor).

In domain $\mathscr{D}_2$ we use the periodicity relation (4.11a) to replace the argument $\lambda + u$ by $\lambda - \pi + u$ (one then has to analytically continue across the $\mathrm{Re}(u) = \lambda - \frac{1}{2}\pi$ boundary). Taking $u_2 = \lambda - u$, $u_4 = \lambda + u$ (or $\lambda - \pi + u$), $x_2 = x_4 = x_0$, $y_2 = y_4 = y_0$, the identity (5.3) becomes

$$I = PR_0^{-2} M e^{-t\lambda \mathscr{H}} P^{-1} \tag{5.6}$$

where R_0 is defined by (5.1) with

$$r_0 = 1 \qquad r_1 = x_0 \qquad r_2 = y_0 \tag{5.7}$$

and

$$\begin{aligned} t &= 2 \qquad \text{in } \mathscr{D}_1 \\ &= 2 - (\pi/\lambda) \qquad \text{in } \mathscr{D}_2 \end{aligned} \tag{5.8}$$

(Thus for the $n = 3$ case we are considering, $\lambda = \pi/7$ and $t = -5$ in $\mathscr{D}_2$.)

From (5.6) it follows that

$$M = R_0^2 e^{t\lambda \mathscr{H}} \tag{5.9}$$

(This result is analagous to eq. (a24) of Ref. 7.)

Now we use the quasi-periodicity relations (4.11), but taking care to remain within $\mathscr{D}_1$ (or $\mathscr{D}_2$). If the nome p of the elliptic function (occurring in (3.1)) is positive, then from (3.10) we can take τ to be pure imaginary. If p is negative, we take $\tau - \frac{1}{2}$ to be pure imaginary. Either way, one can express τ and p in terms of a real positive number s

$$\begin{aligned} p > 0 : \tau = is \qquad & p = e^{-2\pi s} \\ p < 0 : \tau = \tfrac{1}{2} + is \qquad & p = -e^{-2\pi s} \end{aligned} \tag{5.10}$$

In both cases, it follows from (4.11) that $w[u, x, y]$ is proportional to $w[u + 2i\pi s, xe^{-2i\lambda}, ye^{-6i\lambda}]$, so (to within a scalar factor)

$$A(u, x, y) = A(u + 2i\pi s, xe^{-2i\lambda}, ye^{-6i\lambda}) \tag{5.11}$$

and similarly for B, C, D.

Using these periodicity relations in (5.3), we find that the diagonal elements of $\mathscr{H}$ must be of the form

$$\mathscr{H}_{\sigma_1, \ldots, \sigma_m} = s^{-1}[l_{\sigma_1, \ldots, \sigma_m} + \lambda\sigma_1(\sigma_1 + 1)/2\pi] \tag{5.12}$$

where $l_{\sigma_1,\dots,\sigma_m}$ takes only integer values. Assuming these values do not change discontinuously with s, we can obtain them from special limiting cases.

Take $x_1 = x_3 = y_1 = y_3 = 1$, $x_2 = x_4 = x_0$, $y_2 = y_4 = y_0$, $u_1 = u$, $u_3 = 0$, $u_2 + u_4 = t\lambda$. Then from the above arguments $BCD = I$, so (5.3) yields

$$A(u, 1, 1) = Pe^{u\mathscr{H}}P^{-1} \tag{5.13}$$

Let $s \to 0$. From the conjugate modulus identities for the elliptic θ function (eq. 14.2.42 of Ref. 3)

$$\begin{aligned} \theta_1(u) &\sim 2s^{-1/2} \exp\left(-\frac{\pi}{4s} - \frac{u^2}{\pi s}\right) \sinh(u/s) \quad \text{if} \quad p > 0 \\ &\sim i^{1/4}(2/s)^{1/2} \exp\left(-\frac{\pi}{16s} - \frac{u^2}{\pi s}\right) \sinh(u/2s) \quad \text{if} \quad p < 0 \end{aligned} \tag{5.14}$$

provided $-\pi/2 < \mathrm{Re}(u) < \pi/2$.

Now consider the limit when $u, s \to 0$, the ratio u/s remaining finite. Define (for a, b, d integers)

$$\begin{aligned} h(d, b) &= |b - d|/2 \quad \text{if} \quad b + d \text{ is even} \\ &= (b + d + 1)/2 \quad \text{if} \quad b + d \text{ is odd} \end{aligned} \tag{5.15}$$

$$\begin{aligned} p > 0 : \rho &= e^{2u(\lambda - u)/\pi s} \\ g(a) &= a(a + 1)/14 \\ H(d, a, b) &= -h(d, b) \end{aligned} \tag{5.16a}$$

$$\begin{aligned} p < 0 : \rho &= e^{-u(2u + 5\lambda)/\pi s} \\ g(a) &= a(a + 1)/14 \\ H(d, a, b) &= a \end{aligned} \tag{5.16b}$$

Then by using (5.14) in (4.1) (with $x = y = 1$), we find that $\rho^{-1}w(a, b, c, d)$ tends to a limit. This limit is zero if $a \neq c$, nonzero if $a = c$, and is given by

$$\rho^{-1}w(a, b, c, d) = \delta(a, c) \exp\{(u/s)[g(b) + g(d) - g(a) - g(c) + H(d, a, b)]\} \tag{5.17}$$

The factor $\delta(a, c)$ ensures that the lower-right corner transfer matrix A is diagonal. From its definition,[2,3,13] using (5.17), one readily finds that (to within a scalar normalization factor)

$$A_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} = \delta(\boldsymbol{\sigma}, \boldsymbol{\sigma}') \exp\left\{\frac{u}{s}\left[g(\sigma_1) + \sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})\right]\right\} \tag{5.18}$$

Here σ_{m+1} and σ_{m+2} are fixed boundary spins. For an ordered state, they should be set at the appropriate ground-state values.

Thus in the small u, s limit, A is diagonal, so from (5.13) we can take $P = I$ and the diagonal elements of $\mathscr{H}$ to be

$$\mathscr{H}_{\sigma_1,\dots,\sigma_m} = s^{-1}\left[g(\sigma_1) + \sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})\right] \tag{5.19}$$

Since $H(a, b, c)$ and $g_a - a(a+1)/14$ are integer functions, this result is indeed of the form (5.12). Assuming that the integer function $l_{\sigma_1,\dots,\sigma_m}$ in (5.12) does not change discontinuously with s (within a given u domain and for a given sign of p), it follows that (5.19) must be true for all real positive s.

From (5.9), the elements $M(\sigma_1,\dots, \sigma_m)$ of the diagonal matrix M (the diagonal form of $ABCD$) are therefore given by

$$M(\sigma_1,\dots, \sigma_m) = r_{\sigma_1}^2 \exp\left\{\frac{t\lambda}{s}\left[g(\sigma_1) + \sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})\right]\right\} \tag{5.20}$$

where r_a is given by (5.7) and (4.2) and t by (5.8). This is the $M(\sigma_1,\dots, \sigma_m)$ needed in (2.5) and (2.6).

Inversion Relations for z

The partition function per site z defined by (2.2), can be calculated quite simply by the inversion relation technique.[11,14] Regard the nome p as given; then, from (4.1), the Boltzmann weights w are functions of u, x, y. So, therefore, is z, except that from (4.12) and (2.1) the x and y factors cancel out of the partition function, so z is a function only of u. The inversion relation (4.8) implies that

$$z(u)\, z(-u) = (\lambda + u)(\lambda - u)(2\lambda + u)(2\lambda - u)/[(\lambda)^2(2\lambda)^2] \tag{5.21}$$

where $\theta_1(u)$ has again been abbreviated to (u), u is to be regarded as confined to one or other of the domains (5.2), $z(u)$ being the partition function per site in this domain, and $z(-u)$ as its analytic continuation through $u = 0$.

We need a second relation in order to determine $z(u)$. For the domain $\mathscr{D}_1$, we can simply use the rotation symmetry (4.7), which implies that

$$z(u) = z(\lambda - u) \qquad u \in \mathscr{D}_1 \tag{5.22}$$

For domain $\mathscr{D}_2$ we have to work a little harder. First we apply the rotation symmetry (4.7) to both functions w in (4.8). Then we use the periodicity property (4.11a) to replace the argument $\lambda + u$ of the second function by $\lambda - \pi + u$. Finally we replace u throughout by $\lambda - u$. The result is a second inversion relation for w, which implies that

$$z(u)\, z(t\lambda - u) = (2\lambda - u)(u)(3\lambda - u)(\lambda + u)/[(\lambda)^2(2\lambda)^2] \qquad (5.23)$$

t being given by (5.8). (This relation is actually true for both $\mathscr{D}_1$ and $\mathscr{D}_2$.)

As in deriving (5.12), the quasi-periodicity relations (4.11), together with (5.10), can be used to relate w for two values of u differing by $2i\pi s$. (Hence if one lies in $\mathscr{D}_1$, or $\mathscr{D}_2$, then so does the other.) It follows that

$$z(u) = p^4 e^{4i(2u - \lambda)} z(u + 2i\pi s) \qquad (5.24)$$

If one assumes (as seems likely) that $\ln(z(u)$ is analytic inside the appropriate domain ($\mathscr{D}_1$ or $\mathscr{D}_2$) and on its boundaries, then it is determined by (5.21), (5.23), and (5.24). We do not pursue this calculation further in this section: it is discussed in greater generality in Section 6.

6. CONJECTURED GENERALIZATION TO ARBITRARY *n*

In Section 5 we have considered the $n = 3$ case when w is given by (4.1). We can also consider the $n = 2$ case (hard hexagons), when w is given by (3.3). Very little changes: the properties in (3.2), (3.5)–(3.13) are almost the same as the corresponding $n = 3$ properties (4.3), (4.7)–(4.12). The only differences are that $\lambda = \pi/5$ instead of $\pi/7$; a, b, c, d can no longer have value 2, so the y arguments of w disappear; the factor $(2\lambda + u)(2\lambda - u)/(2\lambda)^2$ in (4.8) no longer occurs; the scalar factors multiplying w in the (4.11) are the squares of those in (3.12).

As a result, the equations of Sections 5 are hardly altered for the $n = 2$ case. The y arguments, and references to r_2, disappear. In particular, (5.1), (5.2), (5.6), (5.8), (5.9), (5.10), (5.12), (5.15), and (5.17)–(5.20) are unchanged.

Some numerical factors change in (5.16) (mainly because of the change in value of π/λ): $2u(\lambda - u)$, $\frac{3}{7}$, $2u + 5\lambda$, and 14 become, respectively, $u(\lambda - u)$, $\frac{2}{5}$, $u + 1\frac{1}{2}\lambda$, and 10. The inversion relations (5.21), (5.23) are corollaries of (4.8), so the factors $(2\lambda + u)(2\lambda - u)/(2\lambda)^2$, $(3\lambda - u)(\lambda + u)/(2\lambda)^2$ disappear. The expressions p^4, $4i(2u - \lambda)$ in (5.24) are replaced by p^2 sign (p), $2i(2u - \lambda)$.

We have *not* solved the star-triangle relation (2.4) for arbitrary values of n in (2.9), but the solved $n = 2$ and $n = 3$ cases do suggest a generalization to arbitrary n. Most of the patterns are fairly obvious, so this extrapolation may not be so dangerous as it appears. At the very least, it provides a convenient way of combining the $n = 2$ and $n = 3$ cases.

For arbitrary $n(\geqslant 2)$, we expect each value of w (considered as a function of u) to be a sum of products of $n-1$ functions $\theta_1(u-k\lambda)$, where k is an integer and

$$\lambda = \pi/(2n+1) \tag{6.1}$$

Note at once that this is different from the eight-vertex solid-on-solid generalization of the hard hexagon model,[7] where the Boltzmann weights contain only a single function $\theta_1(u-k\lambda)$.

The parameters x_0, y_0 are defined by (4.2) or (3.7), and occur in the definition (5.7) of r_a. We generalize this to

$$r_a = \{\theta_1[(n-a)\lambda]/\theta_1(n\lambda)\}^{1/2} \tag{6.2}$$

As well as depending on u (and the spins a, b, c, d), we expect w to depend on $\phi \equiv \{\phi_0, \phi_1, ..., \phi_{n-1}\}$, where $\phi_0 = 1$. Thus we can write it as $w(a, b, c, d \mid u, \phi)$ or simply $w[u, \phi]$. Write $\{1, ..., 1\}$ simply as 1, and define

$$\phi'_a = r_a/\phi_a \tag{6.3}$$

$$\hat{\phi}_a = -\phi_a e^{-a(a+1)i\lambda/2} \tag{6.4}$$

Then (4.7), (4.9), (4.11), and (4.12) generalize to

$$w(a, b, c, d \mid u, \phi) = w(b, c, d, a \mid \lambda - u, \phi') \tag{6.5}$$

$$w(a, b, c, d \mid 0, \phi) = \phi_b \phi_d \delta(a, c)/[\phi_a \phi_c] \tag{6.6}$$

$$w[u, \phi] = (-1)^{n-1} w[u+\pi, \phi] \tag{6.7a}$$

$$= p^{(n-1)/2} e^{i(n-1)(2u-\lambda)} w[u+\tau\pi, \hat{\phi}] \tag{6.7b}$$

$$w(a, b, c, d \mid u, \phi) = \phi_b \phi_d w(a, b, c, d \mid u, 1)/[\phi_a \phi_c] \tag{6.8}$$

We expect $h(d, b)$, $H(d, a, b)$ to be given by (5.15) and (5.16) for all n, and ρ and g_a to be given by

$$p>0 \qquad \rho = e^{(n-1)u(\lambda-u)/\pi s}, \qquad g(a) = a(a+1)/(4n+2) \tag{6.9a}$$

$$p<0 \qquad \rho = e^{-(n-1)u(2u+2n-1)/2\pi s}, \qquad g(a) = a(a+1)/(4n+2) \tag{6.9b}$$

Then $\mathscr{H}$, M must again be given by (5.19) and (5.20). If we define

$$q = e^{t\lambda/s} = e^{\pi t/[(2n+1)s]} \tag{6.10}$$

then it follows that the unnormalized probability $F(\sigma_1)$, defined by (2.6), is

$$F(\sigma_1) = \frac{\theta_1[(n-\sigma_1)\lambda]}{\theta_1(n\lambda)} q^{g(\sigma_1)} X_m(\sigma_1, \sigma_{m+1}, \sigma_{m+2}; q) \tag{6.11}$$

where the function $X_m(a, b, c; q)$ is defined by

$$x_m(\sigma_1, \sigma_{m+1}, \sigma_{m+2}, q) = \sum_{\sigma_2, \ldots, \sigma_m} q^{\sum jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})} \tag{6.12}$$

the inner summation being from $j = 1$ to $j = m$, the outer being over all values of $\sigma_2, \ldots, \sigma_m$ satisfying (1.3), i.e.

$$0 \leqslant \sigma_j + \sigma_{j+1} \leqslant n - 1 \qquad 1 \leqslant j \leqslant m \tag{6.13}$$

As in the hard hexagon[1] and eight-vertex SOS[7] models, the boundary spins $\sigma_{m+1}, \sigma_{m+2}$ should be fixed at their ground-state values.

The parameter t is positive or negative depending on whether u lies in $\mathscr{D}_1$ or $\mathscr{D}_2$ (the domains defined in (5.2)). The definition of $H(a, b, c)$ depends on whether the nome p is positive or negative. Thus there are four regimes to consider

$$\begin{array}{llll} \text{I} & p<0 & u \in \mathscr{D}_2 & t = -(2n-1) \\ \text{II} & p>0 & u \in \mathscr{D}_2 & t = -(2n-1) \\ \text{III} & p>0 & u \in \mathscr{D}_1 & t = 2 \\ \text{IV} & p<0 & u \in \mathscr{D}_1 & t = 2 \end{array} \tag{6.14}$$

In all regimes, we see from (5.10) that

$$|p| = e^{-2\pi s} \tag{6.15}$$

To recapitulate: n, p, u are parameters that are at our disposal; n being a positive integer; p being real and in the interval $(-1, 1)$; and u being real or complex, lying in one of the domains $\mathscr{D}_1$ and $\mathscr{D}_2$ defined by (5.2). Then s is given by (6.15), t by (6.14), and q by (6.10); the functions $\theta_1(u)$ and $g(a)$ are given by (3.1) and (6.9); $h(j)$ and $H(d, a, b)$ are defined by (5.15) and (5.16).

Note that u enters the calculation of $F(\sigma_1)$ only via the value of t. This is a reflection of the fact that the row-to-row transfer matrices of the model (for different values of u) commute: the eigenvectors therefore are independent of u. Since P_r depends only on the eigenvector corresponding to the eigenvalue of largest modulus (Section 7.10 of Ref. 3), P_r is independent of u except when the moduli of the two largest eigenvalues cross (as happens when going from domain $\mathscr{D}_1$ to $\mathscr{D}_2$).

We have assumed that the lattice is infinitely large, which implies that $m \to \infty$. The limiting behavior of $X_m(a, b, c; q)$ depends critically on whether $|q| < 1$ (as in regimes I and II), or $|q| > 1$ (III and IV). In fact the model is critical (in the statistical mechanical sense) when $q = 1$, i.e., when $p = 0$, and we can regard p as a temperature-like "deviation from criticality" variable.

Regime I: Gordon's Identities

In regime I, $0 < q < 1$ and $H(d, a, b) = a$. Thus from (6.12)

$$X_m(\sigma_1, \sigma_{m+1}, \sigma_{m+2}; q) = \sum_{\sigma_2, \ldots, \sigma_m} q^{\sigma_2 + 2\sigma_3 + 3\sigma_4 + \cdots + m\sigma_{m+1}} \tag{6.16}$$

This expression tends to a limit as $m \to \infty$ and becomes precisely the expression that occurs in the algebraic formulation of Gordon's generalization[4,5] of the Rogers–Ramanujan identities. Thus we have indeed found (at least for $n = 3$) a generalization of the hard hexagon model that corresponds to Gordon's theorem (1.1).

We can use (1.1) to express X_∞ in terms of elliptic θ functions. When we substitute the result into (6.11) and (2.7), we find that P_a can be further simplified by using θ-function identities (basically eq. (3.2.25) of Ref. 7). We intend to discuss this working, together with that for regimes II, III, and IV, in a subsequent paper. Here we merely quote the final result. Define functions $E(z, x)$ and $\theta_3(u, y)$, for $|x| < 1$ and $0 < y < 1$, by

$$E(z, x) = \prod_{j=1}^{\infty} (1 - x^{j-1} z)(1 - x^j z^{-1})(1 - x^j) \tag{6.17}$$

$$\theta_3(u, y) = \prod_{j=1}^{\infty} (1 + 2y^{(2j-1)/2} \cos 2u + y^{2j-1})(1 - y^j) \tag{6.18}$$

(taking the positive square root of y), and set

$$x = q^{1/(2n-1)} \tag{6.19}$$

$$p_1 = -(-p)^{1/(2n-1)} \qquad p_2 = (-p)^{2n+1} \tag{6.20}$$

(Thus p and p_1 are negative and p_2 and x are positive.) Then, for $1 \leqslant a \leqslant n$

$$P_{n-a} = \frac{x^{(n-a)^2/2} E(x^a, -x^{(2n+1)/2})\, E(q^a, q^{2n+1})}{E(-x^{1/2}, x)\, E(x^{(2n-1)/2}, x^{4n-2})} \tag{6.21}$$

$$= \frac{2\theta_1(a\lambda, p)\, \theta_1(a\lambda, p_1^2)}{(2n+1)\, \theta_3(0, p_2^2)\, \theta_1(\pi/2, p_1^{2n+1})} \tag{6.22}$$

(In fact these are precisely the corresponding results for the eight-vertex solid-on-solid model: eqs. (3.3.1a) and (3.3.18a) of Ref. 7, with P_a replaced by P_{n-a} and $r = 2n+1$.)

Near criticality, p is small and (6.22) gives

$$P_{n-a} = [4/(2n+1)] \sin^2 a\lambda \{1 - p_1^2(1 + 2\cos 2a\lambda) + \cdots\} \tag{6.23}$$

As in Ref. 7, we can define a critical exponent $\bar{\alpha}$ so that the leading singular correction to P_{n-a} (considered as a function of p) is proportional to $|p|^{1-\bar{\alpha}}$. Then from (6.23) it follows that

$$1 - \bar{\alpha} = 2/(2n-1) \tag{6.24}$$

Partition Function Per Site

Let us define a function $\beta(u)$ by

$$\beta(u) = \prod_{j=1}^{n-1} \frac{\theta_1(j\lambda + u)\,\theta_1(j\lambda - u)}{[\theta_1(j\lambda)]^2} \tag{6.25}$$

Then for arbitrary n we expect the RHS of the inversion relations (3.6) and (4.8) to generalize to $\delta(a, c)\,\beta(u)$. This implies that (5.21) and (5.23) generalize to

$$\begin{aligned} z(u)\,z(-u) &= \beta(u) \\ z(u)\,z(t\lambda - u) &= \beta(\lambda - u) \end{aligned} \tag{6.26}$$

while the quasi-periodicity relation (5.24) becomes

$$z(u) = [\mathrm{sign}(p)]^{n-1}\, p^{2n-2} e^{2i(n-1)(2u-\lambda)} z(u + 2i\pi s) \tag{6.27}$$

We restrict u to lie in either $\mathscr{D}_1$ or $\mathscr{D}_2$. If $\ln z(u)$ is analytic in the chosen domain, and on its boundaries, then it follows from (6.27) that

$$\ln z(u) = \frac{(n-1)\,u(\lambda - u)}{\pi s} + \sum_j c_j e^{ju/s} \tag{6.28}$$

where if p is negative and n is even, the sum is over all half-an-odd-integer values of j; otherwise it is over integer values.

We can calculate the coefficients c_j by taking logarithms of (6.26) and using the formulas

$$\begin{aligned} \ln \theta_1(u, e^{-2\pi s}) = &-\frac{1}{2}\ln s - \frac{(2u-\pi)^2}{4\pi s} \\ &- \sum_{j=1}^{\infty} [e^{-2ju/s} + e^{2j(u-\pi)/s}]/j(1 - e^{-2j\pi/s}) \end{aligned} \tag{6.29a}$$

$$\ln \theta_1(u, -e^{-2\pi s}) = \frac{i\pi}{8} - \frac{1}{2}\ln 2s - \frac{(4u-\pi)^2}{16\pi s} - \sum_{j=1}^{\infty} \{e^{-ju/2s} + (-)^j e^{j(2u-\pi)2s}\}/\{j[1-(-)^j e^{-j\pi/2s}]\} \tag{6.29b}$$

which are valid for $0 < \mathrm{Re}(u) < \pi$ and $0 < \mathrm{Re}(u) < \pi/2$, respectively. We can always map the argument u into these domains by using $\theta_1(u) = \theta_1(\pi - u) = -\theta_1(-u)$.

The simplest case is regime III, when we obtain

$$z(u) = \frac{\theta_1(n\lambda)}{\theta_1(n\lambda + u)} \prod_{j=1}^{n} \frac{\theta_1(2j\lambda - u)}{\theta_1(2j\lambda)} \tag{6.30}$$

(for n odd, the $j = (n+1)/2$ term cancels with the factor before the product).

7. SUMMARY

The main results of this paper are the expressions for the local probability P_r that a given site (deep inside the lattice) contains r particles. They are given by (2.7) and (6.11)–(6.15), together with the definition (5.15)–(5.16) of $H(d, a, b)$.

The next step is to reduce the RHS of (6.12) to a more tractable form, perhaps to a sum of products (and ratios) of elliptic θ functions. In regime I this can be done by using Gordon's generalization of the Rogers–Ramanujan identities. The three other regimes lead to other mathematically interesting identities, which we intend to give in a subsequent paper. They are quite different from those we found for the eight-vertex SOS model.

We should stress that we have only obtained a three-state (0, 1, or 2 particles per site) extension of the hard hexagon model. Thus we only know that (6.11)–(6.15) are true for $n = 3$ and $n = 2$. The generalization to higher values of n is a conjecture. However, in the hard hexagon and eight-vertex SOS models[1,7,15] we know that all sorts of mathematical identities conspire to simplify the final results. If this happens with (6.11)–(6.15), it would be remarkable if there were not a corresponding n-state integrable model.

Note

Since writing this paper, the authors have received preprints from Kuniba, Akutsu, and Wadati[16] giving the same solution of the star-triangle relation, though without a proof, and without the local density results for regimes II and III. They also have obtained the $n=4$ and 5 solutions, which we have used to test our conjectures and to obtain the form in (5.16a) of $H(d, a, b)$.

APPENDIX A

Here we lightly sketch how we obtained the solution (4.1) of the star-triangle relation (2.4).

There are 15 distinct values of $w(a, b, c, d)$, namely those listed in (4.1). Write them sequentially as $\omega_1,..., \omega_{15}$ so, for example, $\omega_4 = w(0111)$. Then four of the 59 equations (2.4) are

$$\begin{aligned}
\omega_4\omega_9'\omega_9'' &= \omega_3\omega_1'\omega_1'' + \omega_5\omega_8'\omega_8'' \\
\omega_1\omega_7'\omega_9'' &= \omega_3\omega_1'\omega_6'' + \omega_5\omega_8'\omega_1'' \\
\omega_2\omega_7'\omega_7'' &= \omega_3\omega_6'\omega_6'' + \omega_5\omega_1'\omega_1'' \\
\omega_9\omega_1'\omega_7'' &= \omega_6\omega_3'\omega_1'' + \omega_1\omega_5'\omega_8''
\end{aligned} \tag{A1}$$

For each equation, another can be obtained by interchanging ω_i with ω_i', for $i=1,..., 15$. Thus each becomes a pair of equations, and the first two-yield four homogeneous linear equations for ω_1'', ω_6'', ω_8'', ω_9''. Their determinant must therefore vanish, giving

$$\omega_5\omega_9\omega_4'\omega_8' - \omega_4\omega_8\omega_5'\omega_9' = \omega_1^2\omega_3'\omega_7' - \omega_3\omega_7\omega_1'^2 \tag{A2}$$

Similarly, the last two equations in (A1) give

$$\omega_3\omega_7\omega_2'\omega_6' - \omega_2\omega_6\omega_3'\omega_7' = \omega_1^2\omega_5'\omega_9' - \omega_5\omega_9\omega_1'^2 \tag{A3}$$

Now regard $\omega_1',..., \omega_{15}'$ as constants and $\omega_1,..., \omega_{15}$ as variables. Then it follows from (A2) that $\omega_5\omega_9$, $\omega_4\omega_8$, ω_1^2, and $\omega_3\omega_7$ are linearly dependent, while from (A3) so are $\omega_3\omega_7$, $\omega_2\omega_6$, ω_1^2, and $\omega_5\omega_9$. This will be so if there exist constants $c_1,..., c_6$ such that

$$\begin{aligned}
\omega_1^2 &= c_1\omega_5\omega_9 + c_2\omega_3\omega_7 \\
\omega_2\omega_6 &= c_3\omega_5\omega_9 + c_4\omega_3\omega_7 \\
\omega_4\omega_8 &= c_5\omega_5\omega_9 + c_6\omega_3\omega_7
\end{aligned} \tag{A4}$$

We require ω_i' to be a special case of ω_i, so that (A4) is true if each ω_i is replaced by ω_i'. Substituting back into (A2) and (A3) gives

$$c_3 = c_2 \qquad c_6 = c_1 \tag{A5}$$

One can obtain an equation similar to (A2) from the first and last equations in (A1). Using (A4), we find that

$$c_1 \omega_1 \omega_6 - c_2 \omega_1 \omega_4 = c_7 \omega_3 \omega_9 \tag{A6}$$

where c_7 is a constant. Similarly, the middle two equations in (A1) give

$$c_1 \omega_1 \omega_2 - c_2 \omega_1 \omega_8 = c_8 \omega_5 \omega_7 \tag{A7}$$

Now we can eliminate ω_2, ω_3, ω_4, ω_5 between the five equations (A4), (A6), (A7) to obtain a biquadratic relation between the variables $\omega_8 \omega_9 / \omega_1 \omega_7$ and $\omega_6 \omega_7 / \omega_1 \omega_9$. This can naturally be parametrized in terms of elliptic functions (section 15.10 of Ref. 3). This quickly leads to a parametrization of $\omega_1, \ldots, \omega_9$. Other equations of the set (2.4) can then be used to obtain $\omega_{10}, \ldots, \omega_{15}$, yielding finally the parametrization (4.1).

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